A probability argument that lays the foundation for the Monte Carlo integrator.

Suppose we want to evaluate the integral $I = \int_a^b g(x)dx$, where g(x) is a real valued function whose antiderivative is not readily available. To see how this problem can be approached by Monte Carlo simulation, let Y be the random variable (b – a)G(X), where X is a continuous random variable uniformly distributed on [a,b]. Then it can be shown that the expected value of Y is given by

$$E(Y) = E[(b - a)g(X)] = (b - a)E[g(X)] = (b - a) \int_a^b g(x)f_X(x)dx = (b - a) \frac{\int_a^b g(x)dx}{b - a} = I$$

where $f_X(x) = \frac{1}{b-a}$ is a probability density function of a uniformly distributed random variable in (a,b). Thus, the problem of evaluating the integral has been reduced to one of estimating the expected value E(Y). In particular, we estimate $E(Y) = I$ by the sample mean

$$\bar{Y}(n) = \frac{\sum_{i=1}^{n} Y_i}{n} = (b-a)\frac{\sum_{i=1}^{n} g(X_i)}{n}$$

where $X_1, X_2, \ldots, X_n$ are independent identically distributed uniform random variables in (a,b). We think of $\bar{Y}(n)$ as an estimate of the area of the rectangle with base length b – a and height $\frac{\int_a^b g(x)dx}{b-a}$, which is the average value of g(x) over the interval [a,b]. In addition it can be shown that $E[\bar{Y}(n)] = I$, that is, $\bar{Y}(n)$ is an unbiased estimator of I, and $Var[\bar{Y}(n)] = \frac{Var(Y)}{n}$. Since Var(Y) is a fixed number, it follows that $\bar{Y}(n)$ will be arbitrarily close to I for n sufficiently large.

Adapted from *Simulation Modeling and Analysis*, by Averill M. Law and W. David Kelton, McGraw-Hill, Inc. 1982, pp. 49-50.