Mathematics of the Carpenter's Method[1]

Suppose we want to draw an ellipse using the carpenter's method with a horizontal major axis and a vertical minor axis. For instance an ellipse like in Figure 1.

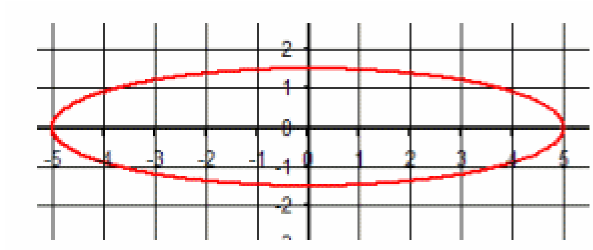

Figure 1.

We make the following identifications:

$$\mathbf{a} = \tfrac{1}{2} \text{ the length of the major axis}$$
$$\mathbf{b} = \tfrac{1}{2} \text{ the length of the minor axis} \qquad (1)$$

with $a \geq b$.

The straight edge used in the carpenter's method is marked as in Figure 2a and positioned as shown in Figure 2b. The action of the method moves point **Q** along the x-axis and point **R** along the y-axis, simultaneously, while point **P** traces out the ellipse.

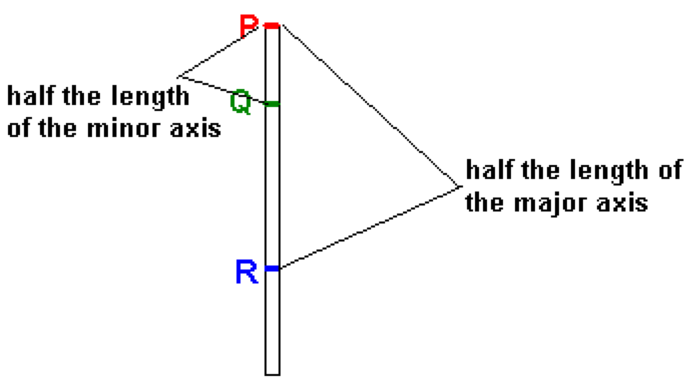

Figure 2a.

[1] Developed by David R. Hill and Sean Comfort, Temple University, Philadelphia, Pa. for Demos with Positive Impact, NSF, DUE 9952306, February 2004.

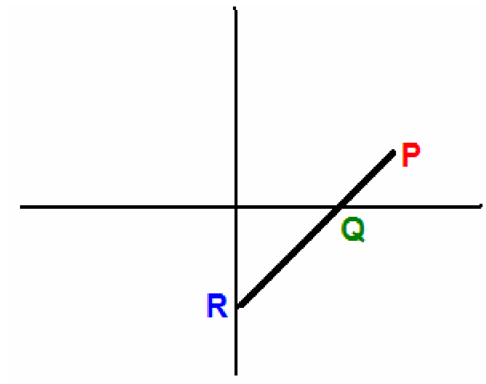

Figure 2b.

By construction

Length of **RQ** = **a** – **b**
Length of **QP** = **b**
Length of **RP** = **a**.

Furthermore we label the coordinates of the three points as follows:

R(0, y), **Q**(x, 0), and **P**(p_1, p_2).

To make things easier to see we enlarge a portion of Figure 2, label features, and include an auxiliary line segment; see Figure 3.

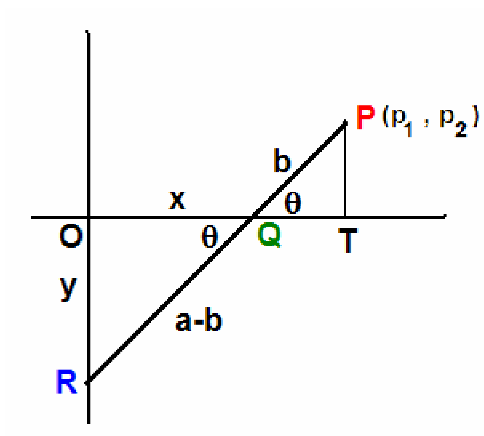

Figure 3.

From ∆**ORQ** we have

$$x = (a - b)\cos(\theta) \tag{2}$$

and since (a – b) ≥ 0 and θ is drawn in the first quadrant,

$$y = -(a-b)\sin(\theta). \tag{3}$$

Thus as the angle θ between the horizontal axis and the straight edge varies we have a parametric representation of the positions of the point **R** and **Q** as given by (3) and (2), respectively. Furthermore from Δ**QTP** we have

$$p_2 = b\sin(\theta). \tag{4}$$

and

$$p_1 = x + b\cos(\theta) = (a-b)\cos(\theta) + b\cos(\theta) = a\cos(\theta). \tag{5}$$

It follows that

$$\frac{p_1^2}{a^2} + \frac{p_2^2}{b^2} = 1,$$

thus the points determined by the carpenter's method do indeed lie on the ellipse with specifications given in (1).

However the parametric representation of the points on the ellipse as given in (4) and (5) has the same *form* as the classic representation by

$$p_1 = a\cos(t),\ p_2 = b\sin(t) \tag{6}$$

where t is an angle at the origin as depicted in Figure 4. Thus we have two

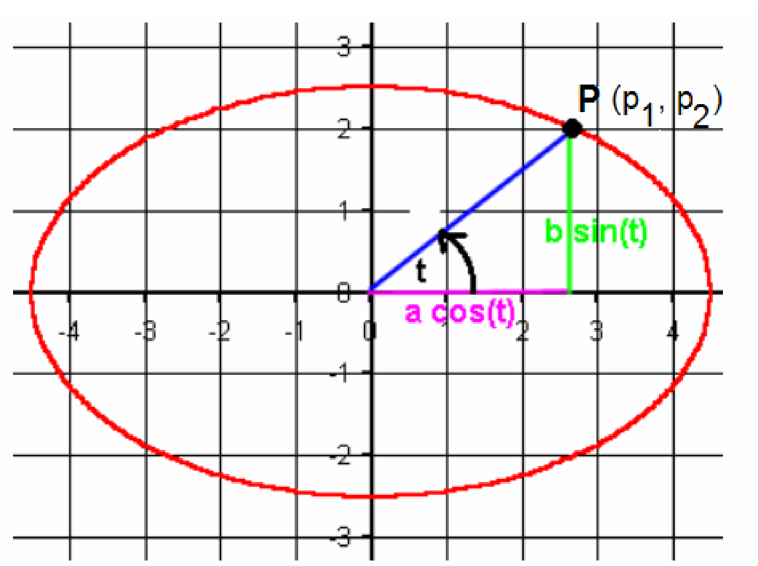

Figure 4.

different parameterizations of an ellipse. We illustrate this in Figure 5. Angle t rotates about the origin, while angle θ changes by the movement of the straightedge from **R** to **P**.

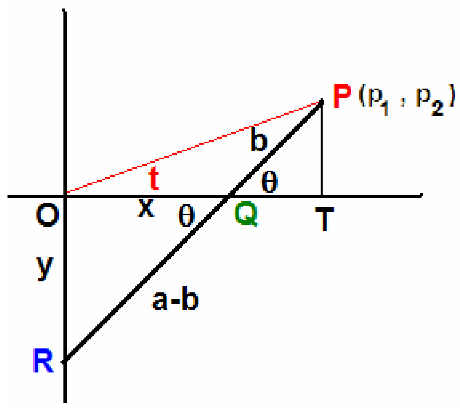

Figure 5.